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Degeneracy of energy levels in a Maslov-quantized perturbed 1:1 resonant oscillator. A quantum counterpart of a Hamiltonian pitchfork bifurcation

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Abstract. It is widely known, in classical theory, that nonlinear Hamiltonian systems depending on several parameters often exhibit bifurcation in their trajectories against the changes in the parameters. A Hamiltonian pitchfork bifurcation in the oscillator's periodic trajectories is a typical example. In this article, to seek a quantum counterpart of that bifurcation, the Maslov quantization is applied to a perturbed 1:1 resonant oscillator in the Birkhoff–Gustavson normal form with two parameters. By using a geometric method the degeneracy of energy levels is found to be taken as a quantum counterpart to that bifurcation. The bifurcation set for the Hamiltonian pitchfork bifurcation in classical theory is viewed as the classical limit of a 'bifurcation set' for the degeneracy of energy levels in quantum theory.

1. Introduction

For many years much attention has been paid to the quantum theory of nonlinear dynamical systems. By applying the semiclassical quantization procedure, for instance, to Hamiltonians expressed in the Birkhoff–Gustavson normal form, various problems are successfully discussed; for example, the Hénon–Heiles system (Swimm and Delos 1979), the diamagnetic Kepler problem (Kuwata et al 1990), etc. In a previous article (Uwano 1989) a quantum study was made for a normal form approximation to the one-parameter Hénon–Heiles Hamiltonian: a degeneracy of energy levels and a 'bifurcation' in the density of eigenfunctions were found to occur at the very parameter value for which the classical system exhibits a bifurcation in its periodic trajectories. In view of this it is plausible to expect that there is a relation between a bifurcation in periodic trajectories in a classical system and a degeneracy of energy levels in the associated quantum system. The aim of this paper is to investigate their relation in the case of a perturbed 1:1 resonant oscillator in the normal form. A close relation between a Hamiltonian pitchfork bifurcation will be found in certain periodic trajectories in the classical system and a degeneracy of energy levels in the Maslov-quantized system. It is shown, indeed, that the Hamiltonian pitchfork bifurcation set for periodic trajectories coincides with the classical limit of a 'bifurcation' set for the degeneracy of energy levels. The perturbed 1:1 resonant oscillator to be dealt with in this *paper is defined to be a Hamiltonian system* $(\mathbf{R}^2 \times \mathbf{R}^2, d\theta_0, H_\alpha)$, where $(\mathbf{R}^2 \times \mathbf{R}^2, d\theta_0)$ is a phase space with the canonical 1-form,

$$\theta_0 = p^T\,\mathrm{d}q = p_1\,\mathrm{d}q_1 + p_2\,\mathrm{d}q_2 \tag{1}$$

expressed in the Cartesian coordinates, $(q, p) \in \mathbf{R}^2 \times \mathbf{R}^2$, and H_α is the Hamiltonian given by

$$H_\alpha = J + \alpha_1 J L_3 + \tfrac{1}{2}\alpha_2 L_2^2 \qquad (\alpha_1 > 0,\ \alpha_2 > 0) \tag{2a}$$

with

$$J = \tfrac{1}{2}\sum_{j=1}^{2}(p_j^2 + q_j^2) \qquad L_1 = q_1 q_2 + p_1 p_2$$
$$L_2 = q_1 p_2 - q_2 p_1 \qquad L_3 = \tfrac{1}{2}(p_1^2 + q_1^2) - \tfrac{1}{2}(p_2^2 + q_2^2). \tag{2b}$$

The α in H_α are parameters taking their values in the region

$$\mathcal{P} = \{\alpha = (\alpha_1, \alpha_2) \in \mathbf{R}^2 | \alpha_1 > 0,\ \alpha_2 > 0\}. \tag{3}$$

This Hamiltonian system can be regarded as a perturbed system for the 1:1 resonant harmonic oscillator, because $H_0 = J$. Moreover, H_α is in the Birkhoff–Gustavson normal form because it commutes with J, i.e. J is a first integral of the perturbed oscillator.

In section 2 the Hamiltonian pitchfork bifurcation of periodic trajectories of the perturbed oscillator is studied against the changes in the parameters α: by using a rotational invariance of the perturbed oscillator (cf Kummer 1976), the half line

$$\mathcal{B} = \{\alpha = (\alpha_1, \alpha_2) | \alpha_1 = \alpha_2,\ \alpha_1 > 0,\ \alpha_2 > 0\} \subset \mathcal{P} \tag{4}$$

is shown to be the Hamiltonian pitchfork bifurcation set.

Section 3 sets up the Maslov quantization. Since H_α and J commute, the perturbed oscillator is a completely integrable Hamiltonian system, so that the Maslov quantization is applicable to it. The first integrals H_α and J determine smooth level sets

$$M_\alpha(h, E) = \left\{(p, q) \in \mathbf{R}^2 \times \mathbf{R}^2 | J(p, q) = h,\ H_\alpha(p, q) = h + h^2 E\right\} \tag{5}$$

with h and E chosen suitably, which are shown to be diffeomorphic (i.e. in smooth one-to-one correspondence) either to a single two-dimensional torus or to a pair of mutually disjoint tori. On each invariant torus a pair of topologically independent non-contractible loops are found. Note that a family of invariant tori stratifies an open-dense domain in $\mathbf{R}^2 \times \mathbf{R}^2$.

In section 4 the Maslov quantization condition is estimated on the loops obtained in section 3. Since the Maslov quantization condition is composed of the action integral and the Maslov index of the loops, this section accordingly is divided in two. In the first part the Maslov indices are calculated, and in the second part the action integrals are shown to be written in terms of certain area integrals.

In section 5 the α-dependence of the degeneracy of energy levels for the Maslov-quantized oscillator is analysed qualitatively by an extensive use of the results obtained in section 4. It turns out that the Hamiltonian pitchfork bifurcation set $\mathcal{B}$ given in (4) for periodic trajectories is identical with the classical limit of a 'bifurcation set' for the degeneracy of energy levels in quantum theory.

Section 6 contains the concluding remarks. The energy level behaviour against the changes in α for the Maslov-quantized oscillator discussed in this paper is compared with that for a quantized oscillator obtained through Robnik's scheme (Robnik 1984): indeed, on computing numerically the energy levels for the 'Robnik-quantized' oscillator, their behaviour against the changes in α agrees with that for the Maslov-quantized oscillator studied in this paper. Further, the bifurcation in the perturbed oscillator studied in this paper is contrasted with a bifurcation in the one-dimensional oscillator with a double-well potential.

Part of this work has been reported in the paper (Uwano 1994) without detailed proof.

2. Hamiltonian pitchfork bifurcation

Let us recall that the harmonic oscillator Hamiltonian J is a first integral for the perturbed oscillator. Hence, every trajectory of the perturbed oscillator is confined on level sets $J^{-1}(h) = \{(q, p) \in \mathbf{R}^2 \times \mathbf{R}^2 | J(q, p) = h\,(>0)\}$. Further, the one-parameter canonical transformation generated by J is a rotation on $\mathbf{R}^2 \times \mathbf{R}^2$,

$$(q, p) \in \mathbf{R}^2 \times \mathbf{R}^2 \longmapsto (R(t)q,\ R(t)p) \in \mathbf{R}^2 \times \mathbf{R}^2 \quad \text{with } R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \tag{6}$$

which leaves the Hamiltonian H_α of the perturbed oscillator invariant. Therefore we can reduce the perturbed oscillator to a lower-dimensional system as follows: since the rotation (6) acts on every $J^{-1}(h)$ without fixed points, a smooth quotient space of $J^{-1}(h)$ is formed by factoring out the rotation. In fact, since the L_j given by (2b) are invariant under the rotation, and since $L_1^2 + L_2^2 + L_3^2 = h^2$ for $(q, p) \in J^{-1}(h)$, the Kustaanheimo–Stiefel (K–S) transformation

$$x = \frac{1}{h}\,(L_1(q, p),\ L_2(q, p),\ L_3(q, p)) \in \mathbf{R}^3 \tag{7}$$

defines the projection π_h of $J^{-1}(h)$ to the quotient space S^2; $\pi_h : J^{-1}(h) \to S^2$. The S^2 is what is called a reduced phase space.

In association with the projection π_h the Hamiltonian equation for the perturbed oscillator is reduced to a Hamiltonian equation on the reduced phase space S^2, which is put in an 'Euler-like' equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -2h\, x \times \operatorname{grad} H^{\mathrm{red}}_{\alpha,h} \qquad (x \in S^2 \subset \mathbf{R}^3) \tag{8}$$

where $\times$ denotes the vector product (cf Kummer 1976), and $H^{\mathrm{red}}_{\alpha,h}$ stands for the reduced Hamiltonian induced from H_α,

$$H^{\mathrm{red}}_{\alpha,h} = h + \alpha_1 h^2 x_3 + \tfrac{1}{2}\alpha_2 h^2 x_2^2. \tag{9}$$

Thus the perturbed oscillator is reduced to the Hamiltonian dynamical system on S^2.

The reduction procedure provides us with the following crucial fact (Kummer 1976, Cushman and Rod 1982):

Fact 2.1. For every equilibrium point of the reduced Hamiltonian equation (8) there exists on $J^{-1}(h)$ a unique periodic trajectory of the perturbed 1:1 resonant oscillator, which projects to the equilibrium point through π_h. Both such a periodic trajectory and its associated equilibrium point share the same stability character.

A bifurcation problem of periodic trajectories allowed to exist by fact 2.1 is then reduced to that of the equilibrium points of the reduced Hamiltonian equation (8). From (8) it follows that all the equilibrium points of (8) are given by the critical points of $H^{\mathrm{red}}_{\alpha,h}$. A straightforward calculation then provides all the equilibrium points as follows:

$$\begin{cases} e_\pm \equiv (0, 0, \pm 1) & \text{if } \alpha_1 \geqslant \alpha_2 \\ e_\pm \quad \text{and} \quad f_\pm \equiv \left(0, \pm\sqrt{1 - \left(\dfrac{\alpha_1}{\alpha_2}\right)^2}, \dfrac{\alpha_1}{\alpha_2}\right) & \text{if } \alpha_1 < \alpha_2. \end{cases} \tag{10}$$

The stability character of these equilibrium points are observed from the Hessian of $H^{\mathrm{red}}_{\alpha,h}$ at the respective points. A calculation results in figure 1, which describes the bifurcation diagram for the equilibrium points (e.p.'s) listed in (10). The full lines stand for the e.p.'s

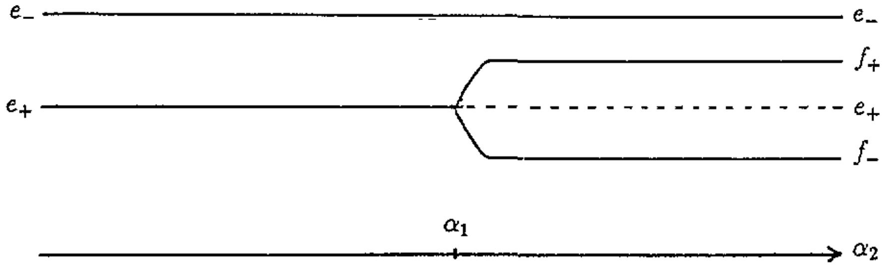

Figure 1. The Hamiltonian pitchfork bifurcation for the e.p.'s of equation (8).

with elliptic stability character, and the broken line for the e.p. with hyperbolic character. Figure 1 shows that the e_+ exhibits a Hamiltonian pitchfork bifurcation (cf Marsden 1992) as α_2 passes through the value α_1. Hence, fact 2.1 implies the following.

Proposition 2.2. For periodic trajectories of the perturbed oscillator allowed to exist by fact 2.1, a Hamiltonian pitchfork bifurcation occurs as $\alpha \in \mathcal{P}$ passes across the bifurcation set $\mathcal{B}$ defined by (4).

3. Invariant tori

Since the perturbed oscillator is a completely integrable Hamiltonian system the Liouville–Arnold theorem (Arnold 1978) implies that the level sets $M_\alpha(h, E)$ already defined by (5) form a family of invariant tori. However, in order that H_α and J are functionally independent we have to restrict $\mathbf{R}^2 \times \mathbf{R}^2$ to the open-dense subset

$$\mathcal{O} = \begin{cases} \{(q, p) \in \mathbf{R}^2 \times \mathbf{R}^2 | J^2 \neq (L_3)^2\} & \text{if} \quad \alpha_1 \geqslant \alpha_2 \\ \{(q, p) \in \mathbf{R}^2 \times \mathbf{R}^2 | J^2 \neq (L_3)^2 \text{ or } H_\alpha \neq J + \alpha_1 J^2\} & \text{if} \quad \alpha_1 < \alpha_2. \end{cases} \tag{11}$$

We wish to show that every smooth $M_\alpha(h, E)$ is indeed diffeomorphic (i.e. in a smooth one-to-one correspondence) either to a two-dimensional torus T^2 or to a pair of T^2. A key is the fact that the $J^{-1}(h)$ is made into an S^1 fibre bundle (Schutz 1980) over S^2, $\pi_h : J^{-1}(h) \to S^2$. The $J^{-1}(h)$ is not the direct product space $S^2 \times S^1$ topologically, but the subset, $U_h \equiv \{(q, p) \mid L_3(q, p) \neq -h\}$, of $J^{-1}(h)$ admits the direct product structure,

$$U_h = \{(q, p) \mid L_3(q, p) \neq -h\} \cong S^1 \times \pi_h(U_h) \quad \text{with} \quad \pi_h(U_h) = \{x \in S^2 \mid x_3 \neq -1\} \tag{12}$$

where S^1 denotes the circle group of the rotation (6). In fact, we can realize (12) by the mapping (cf Uwano 1989),

$$(R(t), x) \in S^1 \times \pi_h(U_h) \longmapsto \sqrt{h}\,(R(t)Q(x),\ R(t)P(x)) \in U_h \qquad (t \in [0, 2\pi]) \tag{13a}$$

where

$$Q(x) = \left(\sqrt{1 + x_3},\ \frac{x_1}{\sqrt{1 + x_3}}\right) \qquad \text{and} \qquad P(x) = \left(0,\ -\frac{x_2}{\sqrt{1 + x_3}}\right). \tag{13b}$$

Since the level set $M_\alpha(h, E)$ within the subset $\mathcal{O}$, (11), is contained in U_h, equation (12) implies that $M_\alpha(h, E)$ is viewed topologically as $S^1 \times \{x \in S^2 \mid H^{\mathrm{red}}_{\alpha,h}(x) = h + h^2 E\}$, where $\{x \in S^2 \mid H^{\mathrm{red}}_{\alpha,h}(x) = h + h^2 E\}$ denotes the level curve of the reduced Hamiltonian $H^{\mathrm{red}}_{\alpha,h}$. Hence we obtain

$$M_\alpha(h, E) \cong S^1 \times \pi_h(M_\alpha(h, E)) \cong S^1 \times \{x \in S^2 \mid H^{\mathrm{red}}_{\alpha,h}(x) = h + h^2 E\}. \tag{14}$$

Thus the topology of $M_\alpha(h, E)$ is determined, depending on the level curve $\{x \in S^2 | H^{\rm red}_{\alpha,h} = h + h^2 E\}$. To look into the level curves it is convenient to introduce the local coordinates (ξ, η) in $S^2 \backslash \{(0, 0, -1)\}$ by

$$x = \left(2\xi\sqrt{1-\xi^2-\eta^2}, 2\eta\sqrt{1-\xi^2-\eta^2}, 1-2(\xi^2+\eta^2)\right) \tag{15a}$$

with the constraint

$$\xi^2 + \eta^2 < 1 \tag{15b}$$

which were used efficiently in Kummer (1976). In terms of (ξ, η) the equation $H^{\rm red}_{\alpha,h} = h + h^2 E$ is put in the form

$$\xi^2 = 1 - \eta^2 - \frac{\alpha_1 + E}{2(\alpha_1 + \alpha_2\eta^2)} \equiv F(\eta) \qquad \text{with} \quad \xi^2 + \eta^2 < 1 \tag{16}$$

so that the level curve $\{x \in S^2 \,|\, H^{\rm red}_{\alpha,h} = h + h^2 E\}$ is mapped to the curve $\{(\xi, \eta) \,|\, \xi^2 = F(\eta),\ \xi^2 + \eta^2 < 1\}$ through the smooth mapping

$$\chi : x \longmapsto (\xi, \eta) = \left(\frac{x_1}{\sqrt{2(1+x_3)}}, \frac{x_2}{\sqrt{2(1+x_3)}}\right) \tag{17}$$

determined by (15). Therefore classifying the topology of $\{x \in S^2 \,|\, H^{\rm red}_{\alpha,h} = h + h^2 E\}$ amounts to classifying the topology of $\chi(\{x \in S^2 \,|\, H^{\rm red}_{\alpha,h} = h + h^2 E\}) = \{(\xi, \eta) \,|\, \xi^2 = F(\eta),\ \xi^2 + \eta^2 < 1\}$ in the (ξ, η)-plane determined by (16). We note here that, in order for equation (16) to determine a non-empty set, the E is required to satisfy

$$\begin{aligned} &-\alpha_1 < E < \alpha_1 && \text{for} \quad \alpha_1 \geqslant \alpha_2 \\ &-\alpha_1 < E < \frac{\alpha_1^2 + \alpha_2^2}{2\alpha_2} && \text{for} \quad \alpha_1 < \alpha_2. \end{aligned} \tag{18}$$

By evaluating (16) along with (18), we have the following.

Lemma 3.1. The non-empty smooth level curves, $\{x \in S^2 \,|\, H^{\rm red}_{\alpha,h} = h + h^2 E\}$, are diffeomorphic to one of the following:

for $\alpha_1 \geqslant \alpha_2$,

$$\{x \in S^2 \,|\, H^{\rm red}_{\alpha,h} = h + h^2 E\} \cong S^1 \qquad \text{if} \quad -\alpha_1 < E < \alpha_1; \tag{19a}$$

for $\alpha_1 < \alpha_2$,

$$\begin{cases} \{x \in S^2 \,|\, H^{\rm red}_{\alpha,h} = h + h^2 E\} \cong S^1 & \text{if} \quad -\alpha_1 < E < \alpha_1 \\ \{x \in S^2 \,|\, H^{\rm red}_{\alpha,h} = h + h^2 E\} \cong S^1 \oplus S^1 & \text{if} \quad \alpha_1 < E < \dfrac{\alpha_1^2 + \alpha_2^2}{2\alpha_2}. \end{cases} \tag{19b}$$

Here $S^1 \oplus S^1$ denotes the disjoint union of two circles, one of which is in the region in S^2 determined by $x_2 > 0$, and the other in the region with $x_2 < 0$.

From (14) and (19) we have the following:

Proposition 3.2. The non-empty smooth level sets $M_\alpha(h, E)$ are diffeomorphic to one of the following:

for $\alpha_1 \geqslant \alpha_2$,

$$M_\alpha(h, E) \cong T^2 \qquad \text{if} \qquad -\alpha_1 < E < \alpha_1; \tag{20a}$$

for $\alpha_1 < \alpha_2$,

$$\begin{cases} M_\alpha(h,E) \cong T^2 & \text{if} \quad -\alpha_1 < E < \alpha_1 \\ M_\alpha(h,E) \cong T^2 \oplus T^2 & \text{if} \quad \alpha_1 < E < \dfrac{\alpha_1^2+\alpha_2^2}{2\alpha_2} \end{cases} \tag{20b}$$

where $T^2 \cong S^1 \times S^1$. One T^2 of the disjoint union $T^2 \oplus T^2$ is in the region $\{(q,p) \mid L_2(q,p) > 0\}$, and the other in $\{(q,p) \mid L_2(q,p) < 0\}$.

We wish here to mention what happens in the case of $E = \alpha_1$ with $\alpha_1 < \alpha_2$, which is listed in neither (19*b*) nor (20*b*). In this case it happens that $L_3(q,p) = h$, and the level curve $\{x \in S^2 \mid H^{\text{red}}_{\alpha,h} = h + h^2 E\}$ looks like a '*figure of eight*', which consists of two circles glued at a point. Part of the figure of eight is viewed topologically as a circle,

$$\{H^{\text{red}}_{\alpha,h} = h + h^2\alpha_1\} \cap \{x_2 \geqslant 0\} \cong \{H^{\text{red}}_{\alpha,h} = h + h^2\alpha_1\} \cap \{x_2 \leqslant 0\} \cong S^1. \tag{21a}$$

Then $M_\alpha(h,\alpha_1)$ takes the form of $S^1 \times$ '*figure of eight*'. Since the $M_\alpha(h,\alpha_1)$ may be viewed as a limit of $T^2 \oplus T^2$ given in (20b), $M_\alpha(h,\alpha_1) \cap \{L_2(q,p) \geqslant 0\}$ and $M_\alpha(h,\alpha_1) \cap \{L_2(q,p) \leqslant 0\}$ will be referred to also as '*invariant tori*',

$$M_\alpha(h,\alpha_1) \cap \{L_2(q,p) \geqslant 0\} \cong M_\alpha(h,\alpha_1) \cap \{L_2(q,p) \leqslant 0\} \cong T^2 \tag{21b}$$

which will play a critical role in the following sections.

The above discussion also provides a pair of topologically independent non-contractible loops of the invariant tori. One of the non-contractible loops is, of course, the factor space, S^1, coming from the rotation (6). The other loop comes from the level curves (19).

Proposition 3.3. For a given invariant torus one of the non-contractible loops is realized as the orbit in T^2 generated by the rotation (6): on choosing a point, say $(\tilde{q}, \tilde{p})$ in T^2, the loop, denoted by $c^{(1)}$, is given by

$$c^{(1)} = \{(q,p) \in T^2 | (q,p) = (R(t)\tilde{q},\ R(t)\tilde{p}), t \in [0, 2\pi]\}. \tag{22}$$

Note that the point $(\tilde{q}, \tilde{p})$ can be chosen arbitrarily as long as it belongs to the invariant torus under consideration. The loop $c^{(1)}$ will be referred to as a 'type-I loop' henceforth.

Proposition 3.4. Another non-contractible loop, denoted by $c^{(2)}$, is given by

$$c^{(2)} = \{(q,p) \in T^2 \mid (q,p) = \sqrt{h}(Q(x), P(x)), x \in \pi_h(T^2)\} \tag{23}$$

where $Q(x)$ and $P(x)$ are defined by (13*b*), and the projection π_h by (7). The $\pi_h(T^2)$ of T^2 is either of the loops listed in (19). The loop $c^{(2)}$ will be referred to as the 'type-II loop' henceforth.

Remark. In the case of the 'singular' invariant tori, (21*b*), propositions 3.3 and 3.4 also hold true.

4. The Maslov quantization condition

So far we have found the pair of topologically independent non-contractible loops in the invariant tori. We are now in a position to estimate the Maslov quantization condition for the perturbed oscillator. Note here that the Maslov quantization is also referred to as the torus quantization or as the EBK (Einstein–Brillouin–Keller) quantization (Gutzwiller 1993). According to the Maslov quantization condition (Abraham and Marsden 1978) an invariant torus is Maslov quantizable if the integral condition

$$\frac{1}{2\pi\hbar}\oint_c p^T\,\mathrm{d}q - \tfrac{1}{4}\mathcal{M}(c) = \text{an integer} \qquad (\hbar = (\text{Planck's constant})/2\pi) \tag{24}$$

is valid for every (smooth) loop c in the invariant torus. The first term in the left-hand side of (24) is the action integral and the second is one quarter of the Maslov index. In practice, owing to Stokes' theorem (Schutz 1980), we have only to check (24) for the pair of non-contractible loops $c^{(1)}$ and $c^{(2)}$ (see (22) and (23)). We note here that in Weinstein (1974) the Maslov quantizable invariant tori are introduced as those which represent 'quasi-classical states' of a given Hamiltonian system.

4.1. *The Maslov indices*

The complete integrability of our perturbed oscillator enables us to describe the Maslov indices in the form of line integral using H_α and J (Arnold 1963, Yoshioka 1986). The calculation of the integral (appendix A) provides the following:

Lemma 4.1. For the loops $c^{(1)}$ and $c^{(2)}$, given by (22) and (23), the Maslov indices are calculated, except for the case $E = \alpha_1$ with $\alpha_1 < \alpha_2$, to be

$$\mathcal{M}(c^{(1)}) = 4 \qquad \text{and} \qquad \mathcal{M}(c^{(2)}) = -2. \tag{25}$$

Remark. As is pointed out in section 3, the tori $M_\alpha(h, \alpha_1) \cap \{L_2(q,p) \geqslant 0\}$ and $M_\alpha(h, \alpha_1) \cap \{L_2(q,p) \leqslant 0\}$ have singularities for $E = \alpha_1$, so that the Maslov indices for the loops $c^{(1)}$ and $c^{(2)}$ for these tori are not defined well. However, since we wish to deal with these 'singular' invariant tori together with the regular ones, we will try later to associate Maslov indices to these singular loops.

4.2. *The action integrals*

We compute the action integral in the following. For this purpose we put $p^T\,\mathrm{d}q$ into the form

$$p^T\,\mathrm{d}q = \tfrac{1}{2}(p^T\,\mathrm{d}q - q^T\,\mathrm{d}p) + \mathrm{d}\left(\tfrac{1}{2}p^T q\right). \tag{26}$$

Hence Stokes' theorem (Schutz 1980) shows that

$$\frac{1}{2\pi\hbar}\oint_c p^T\,\mathrm{d}q = \frac{1}{2\pi\hbar}\oint_c \tfrac{1}{2}(p^T\,\mathrm{d}q - q^T\,\mathrm{d}p). \tag{27}$$

The use of (27) makes it easy to calculate the action integral. For the type-I loop $c^{(1)}$ equation (27) yields the following.

Lemma 4.2. For the type-I loop $c^{(1)}$, given by (22), the action integral is calculated to be

$$\frac{1}{2\pi\hbar}\oint_{c^{(1)}} p^T\,\mathrm{d}q = \frac{h}{\hbar}. \tag{28}$$

We turn to the action integral for the type-II loop:

Lemma 4.3. For the type-II loop $c^{(2)}$, given by (23), the action integral is put into the form,

$$\frac{1}{2\pi\hbar}\oint_{c^{(2)}} p^T\,\mathrm{d}q = -\frac{h}{\pi\hbar}\,\mathrm{Area}\,(\tilde{c}^{(2)}) \tag{29}$$

where $\tilde{c}^{(2)}$ denotes the loop $\chi(c^{(2)})$ (see (17)) in the (ξ, η)-plane, and Area $(\tilde{c}^{(2)})$ the area of the region enclosed by $\tilde{c}^{(2)}$ oriented counter-clockwise.

Proof. For $c^{(2)}$ of (23), equation (27) is written as

$$\frac{1}{2\pi\hbar}\oint_{c^{(2)}} p^T\,\mathrm{d}q = \frac{1}{2\pi\hbar}\oint_{c^{(2)}} \tfrac{1}{2}(p^T\,\mathrm{d}q - q^T\,\mathrm{d}p)$$

$$= -\frac{h}{2\pi\hbar}\oint_{\tilde{c}^{(2)}}(\xi\,\mathrm{d}\eta - \eta\,\mathrm{d}\xi) = -\frac{h}{\pi\hbar}\mathrm{Area}\,(\tilde{c}^{(2)}). \tag{30}$$

In the last equality in (30) Green's theorem is applied with the counter-clockwise orientation of $\tilde{c}^{(2)}$ taken into account. □

In view of lemma 4.3 we investigate Area $(\tilde{c}^{(2)})$ in order to estimate the action integral for the type-II loop $c^{(2)}$. It will then be convenient to label the type-II loops $c^{(2)}$ by the triple $(h, E; \alpha)$, because of proposition 3.2: if $\alpha_1 \leqslant E < (\alpha_1^2 + \alpha_2^2)/2\alpha_2$ with $\alpha_1 < \alpha_2$, the invariant tori for $(h, E; \alpha)$ arise in pairs, so that we denote by $c_+^{(2)}(h, E; \alpha)$ and $c_-^{(2)}(h, E; \alpha)$ the type-II loops for the tori $M_\alpha(h, E) \cap \{L_2(q, p) \geqslant 0\}$ and $M_\alpha(h, E) \cap \{L_2(q, p) \leqslant 0\}$, respectively. In contrast to this, if $-\alpha_1 < E < \alpha_1$ a single type-II loop arises for $(h, E; \alpha)$, which will be denoted by $c_s^{(2)}(h, E; \alpha)$, with s indicating 'single'.

We first calculate Area $(\tilde{c}^{(2)})$ in the case of $E = \alpha_1$. A straightforward calculation yields (see appendix B),

$$\mathrm{Area}\,(\tilde{c}_\pm^{(2)}(h, \alpha_1; \alpha)) = \frac{\pi}{2} - \arcsin\left(\sqrt{\frac{\alpha_1}{\alpha_2}}\right) - \frac{\alpha_1}{\alpha_2}\sqrt{\frac{\alpha_2}{\alpha_1} - 1} \tag{31}$$

where $0 < \arcsin(\sqrt{\alpha_1/\alpha_2}) < \pi/2$. The (31) will be of great help in the succeeding discussions because Area $(\tilde{c}_\pm^{(2)}(h, \alpha_1; \alpha))$ will play a critical role.

In the case of $E \neq \alpha_1$ the area integrals Area $(\tilde{c}_\sigma^{(2)}(h, E; \alpha))$ $(\sigma = s, \pm)$ are expressed in terms of elliptic integrals after calculation (appendix B). Indeed, for $\tilde{c}_\pm^{(2)}(h, E; \alpha)$ we have

$$\mathrm{Area}\,(\tilde{c}_\pm^{(2)}(E, \alpha)) = 2\left[-\frac{\alpha_2}{\alpha_1}\int_0^{w_c}\left(\frac{1}{1+w^2}\right)^2\frac{\mathrm{d}w}{\sqrt{G(w)}} + \frac{\alpha_2 + E}{\alpha_1}\int_0^{w_c}\frac{1}{1+w^2}\frac{\mathrm{d}w}{\sqrt{G(w)}}\right.$$
$$\left. - \frac{E}{\alpha_1}\int_0^{w_c}\frac{\mathrm{d}w}{\sqrt{G(w)}}\right] \tag{32}$$

where $G(w)$ is the quartic polynomial,

$$G(w) = w^4 + 2\left(1 - \frac{E\alpha_2}{\alpha_1^2}\right)w^2 + \left(1 - \frac{E\alpha_2}{\alpha_1^2} + \frac{\alpha_2^2}{\alpha_1^2}\right) \tag{33}$$

and $w_c\,(>0)$ is the zero of $G(w)$ put in the form

$$w_c = \frac{1}{\alpha_1}\left[E\alpha_2 - \alpha_1^2 - \alpha_2\sqrt{E^2 - \alpha_1^2}\right]^{1/2}. \tag{34}$$

For the loop $\tilde{c}_s^{(2)}(h, E, \alpha)$ we have

$$\mathrm{Area}\,(\tilde{c}_s^{(2)}(h, E, \alpha)) = \frac{\pi}{2} + 4\left[-\frac{\alpha_2}{\alpha_1}\int_0^{\infty}\left(\frac{1}{1+w^2}\right)^2\frac{\mathrm{d}w}{\sqrt{G(w)}}\right.$$
$$\left. + \frac{\alpha_2 + E}{\alpha_1}\int_0^{\infty}\frac{1}{1+w^2}\frac{\mathrm{d}w}{\sqrt{G(w)}} - \frac{E}{\alpha_1}\int_0^{\infty}\frac{\mathrm{d}w}{\sqrt{G(w)}}\right]. \tag{35}$$

However, it is hard to derive the quantized energy levels E even if we use the expressions (32)–(35). Hence, our discussion will centre on the qualitative behaviour of the quantized energy levels henceforth. In what follows we present several properties of the area functions Area $(\tilde{c}_\sigma^{(2)}(h, E; \alpha))$ $(\sigma = s, \pm)$ (see appendix C for their proofs).

Lemma 4.4. The loop $\tilde{c}_\sigma^{(2)}(h, E; \alpha)$ in the (ξ, η)-plane given as $\chi(c_\sigma^{(2)}(h, E; \alpha))$ (see (17)) is independent of h, and so is the area integral Area $(\tilde{c}_\sigma^{(2)}(h, E; \alpha))$ $(\sigma = s, \pm)$.

Owing to lemma 4.4 the parameter h is omitted in the area integrals Area $(\tilde{c}_\sigma^{(2)}(E; \alpha))$ $(\sigma = s, \pm)$ henceforth.

Lemma 4.5. Let α be fixed. Then the area integrals Area $(\tilde{c}_\pm^{(2)}(E; \alpha))$ are decreasing continuous functions of E as E increases from α_1 to $(\alpha_1^2 + \alpha_2^2)/2\alpha_2$.

We further obtain a formula from (32):

Lemma 4.6. In the case of $\alpha_1 \leqslant E < (\alpha_1^2 + \alpha_2^2)/2\alpha_2$ with $\alpha_1 < \alpha_2$, the area integrals for $\tilde{c}_+^{(2)}(E; \alpha)$ and for $\tilde{c}_-^{(2)}(E; \alpha)$ are equal,

$$\text{Area}\,(\tilde{c}_+^{(2)}(E; \alpha)) \equiv \text{Area}\,(\tilde{c}_-^{(2)}(E; \alpha)). \tag{36}$$

5. The degeneracy of energy levels and its bifurcation set

In this section we discuss a degeneracy of energy levels for the Maslov-quantized perturbed oscillator qualitatively by using the results obtained in section 4. We first mention that the Maslov quantization condition (24) for the type-I loop provides quantized values h of J. In fact, inserting (25) and (28) into the quantization condition (24) we have the following result, which agrees with the usual one for the harmonic oscillator Hamiltonian J.

Lemma 5.1. The value h of the harmonic oscillator Hamiltonian J is quantized as

$$h = \hbar(N + 1) \quad (N = 0, 1, 2, \ldots). \tag{37}$$

We consider, in turn, what results from the quantization condition for the type-II loop. The first major result comes from lemma 4.6: for $\alpha_1 < E < (\alpha_1^2 + \alpha_2^2)/2\alpha_2$ with $\alpha_1 < \alpha_2$, equations (24), (25), (29), and (36) taken together provide the equality

$$\frac{1}{2\pi\hbar} \oint_{c_+^{(2)}(h,E;\alpha)} p^T \mathrm{d}q - \tfrac{1}{4}\mathcal{M}(c_+^{(2)}(h, E; \alpha)) = \frac{1}{2\pi\hbar} \oint_{c_-^{(2)}(h,E;\alpha)} p^T \mathrm{d}q - \tfrac{1}{4}\mathcal{M}(c_-^{(2)}(h, E; \alpha)) \tag{38}$$

which implies that if the invariant torus $M_\alpha(h, E) \cap \{L_2(q, p) \geqslant 0\}$ for $c_+^{(2)}(h, E; \alpha)$ satisfies the Maslov quantization condition, then so does $M_\alpha(h, E) \cap \{L_2(q, p) \leqslant 0\}$ for $c_-^{(2)}(h, E; \alpha)$, and vice versa. Put another way, if an energy level E with $\alpha_1 < E < (\alpha_1^2 + \alpha_2^2)/2\alpha_2$ is admissible by the Maslov quantization condition, then there exist two different Maslov quantizable invariant tori of the same energy level E. In contrast to this, if $-\alpha_1 < E < \alpha_1$, proposition 3.2 ensures that there exists only one Maslov quantizable torus, $M_\alpha(h, E)$. To summarize, we have the following.

Proposition 5.2. If an energy level E with $\alpha_1 < E < (\alpha_1^2 + \alpha_2^2)/2\alpha_2$ is admissible by the Maslov quantization condition (24), it must be doubly degenerate.

We wish to study the α-dependence of this degeneracy. According to lemma 4.5 the Maslov quantization condition for the type-II loop $c_\pm^{(2)}(h, E; \alpha)$ is estimated, by using (30) and (31), as follows:

$$\begin{aligned} \tfrac{1}{2} &> \frac{1}{2\pi\hbar} \oint_{c_\pm^{(2)}(h,E;\alpha)} p^T \mathrm{d}q - \tfrac{1}{4}\mathcal{M}(c_\pm^{(2)}(h, E; \alpha)) \\ &> -\frac{h}{\pi\hbar}\,\text{Area}\,(\tilde{c}_\pm^{(2)}(\alpha_1; \alpha)) + \tfrac{1}{2} > -\frac{h}{2\hbar} + \tfrac{1}{2}. \end{aligned} \tag{39}$$

If h is quantized as $h = (N+1)\hbar$ $(N = 0, 1, \ldots)$ (see (37)) the right-hand side of (39) becomes $-(h/2\hbar) + (1/2) = -(N/2)$, so that admissible integers have to be 0 to $-[(N-1)/2]$, where $[(N-1)/2]$ denotes the integer part of $(N-1)/2$. The middle part of (39) along with (31) then implies the following.

Proposition 5.3. Assume that the h is quantized as $h = \hbar(N+1)$ ($N \geqslant 0$; a non-negative integer). If the inequality

$$-\nu - 1 \leqslant -\frac{N+1}{\pi}\,\mathrm{Area}\,(\tilde{c}_{\pm}^{(2)}(\alpha_1;\alpha)) + \tfrac{1}{2} < -\nu \tag{40}$$

holds for $\nu = 0, 1, \ldots, [(N-1)/2]$, the Maslov quantization condition has to be

$$-\frac{N+1}{\pi}\mathrm{Area}\,(\tilde{c}_{\pm}^{(2)}(E;\alpha)) + \tfrac{1}{2} = 0, -1, \ldots, -\nu \tag{41}$$

so that there exist $\nu+1$ doubly-degenerate energy levels subject to $\alpha_1 < E < (\alpha_1^2+\alpha_2^2)/2\alpha_2$.

Proof. If the condition (40) holds, the inequality (39) clearly implies (41). Then there exists a unique energy level E which satisfies $\mathrm{Area}\,(\tilde{c}_{\pm}^{(2)}(E;\alpha)) = \pi(j+\frac{1}{2})/(N+1)$ $(j = 0, 1, \ldots, \nu)$, on account of lemma 4.5. Thus, we find $\nu+1$ doubly-degenerate energy levels.

So far we have Maslov-quantized regular tori. How could we deal with the singular tori (see (21*b*)) to which the Maslov quantization rule does not apply strictly? We wish to 'quantize' the singular tori in some sense in spite of their singularity: if they were not 'quantizable', the energy level $E = \alpha_1$ would become prohibited, and therefore the continuity of the energy levels in α would break down. A way to settle this 'quantization' problem is to think of the singular tori as the limit of the regular tori (see (20*b*)), as $E > \alpha_1$ tends to α_1 (see lemma 4.5). Under this continuity hypothesis we could assume that the singular tori share the same Maslov index with the regular tori, i.e. we might have

$$\mathcal{M}(c_{\pm}^{(2)}(h, \alpha_1; \alpha)) = -2. \tag{42}$$

Let us recall the quantization condition (41) for the regular tori. Then, under the above hypothesis with lemma 4.5 and (42), a 'quantization' condition for the singular tori could be

$$\begin{aligned} &-\frac{N+1}{\pi}\,\mathrm{Area}\,(\tilde{c}_{\pm}^{(2)}(\alpha_1;\alpha)) + \tfrac{1}{2} \\ &\qquad = -\frac{N+1}{\pi}\left[\frac{\pi}{2} - \arcsin\left(\sqrt{\frac{\alpha_1}{\alpha_2}}\right) - \frac{\alpha_1}{\alpha_2}\sqrt{\frac{\alpha_2}{\alpha_1} - 1}\right] + \frac{1}{2} = -\nu \end{aligned} \tag{43}$$

for $\nu = 0, 1, \ldots, [\frac{1}{2}(N-1)]$. Hence when the condition (43) is satisfied for $E = \alpha_1$ the value $E = \alpha_1$ could be an admissible doubly-degenerate energy level. The discussion above and proposition 5.3 taken together imply the following:

Theorem 5.4. Assume that h is quantized as $h = \hbar(N+1)$ (N: a non-negative integer). There exist $\nu+1$ doubly-degenerate energy levels, if the inequality

$$-\nu - 1 < -\frac{N+1}{\pi}\,\mathrm{Area}\,(\tilde{c}_{\pm}^{(2)}(\alpha_1;\alpha)) + \tfrac{1}{2} \leqslant -\nu \qquad \nu = 0, 1, \ldots, [\tfrac{1}{2}(N-1)] \tag{44}$$

holds, otherwise no degeneracy occurs.

Note that if the equality holds in (44) one of $\nu+1$ energy levels corresponds to the singular tori, and the remaining ν energy levels to regular tori. If the equality does not hold, all of $\nu+1$ energy levels are for the regular tori.

The condition (43) with $\nu = 0$ determines when a degenerate energy level takes place if $h = \hbar(N+1)$ is fixed. In fact, equation (43) with $\nu = 0$ is an equation of α_1/α_2. On denoting by r_N the unique solution of (43) with $\nu = 0$, the equation $\alpha_1/\alpha_2 = r_N$ determines a straight line in $\mathcal{P}$, which will be denoted by ℓ_N henceforth. Since the middle terms of (43) can be easily shown to be an increasing function in α_1/α_2, we have the following:

Lemma 5.5. The degeneracy of energy levels allowed to exist by theorem 5.4 takes place if and only if α satisfies $\alpha_1/\alpha_2 \leqslant r_N$.

Hence ℓ_N can be thought of as a 'bifurcation set'; the degeneracy of energy levels takes place or not according to whether α lies above or on the line ℓ_N, or below the line ℓ_N (see figure 2).

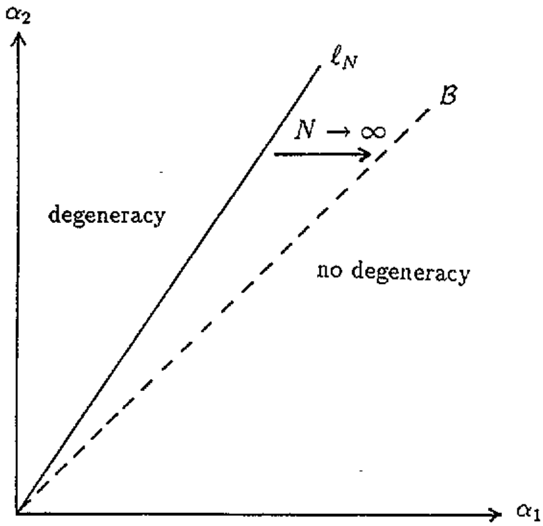

Figure 2. The bifurcation set ℓ_N for the degeneracy of energy levels.

We are to show that the bifurcation set ℓ_N actually tends to the bifurcation set $\mathcal{B}$ (see (4)) in classical theory when $N \to \infty$ for ℓ_N. To this end we need to know the behaviour of the bifurcation set ℓ_N against the non-negative integer N. Since the solution $\alpha_1/\alpha_2 = r_N$ of (43) with $\nu = 0$ is the slope of ℓ_N, it tends to 1 as N tends to infinity, as is easily seen from (43).

Lemma 5.6. As the integer N in (37) tends to infinity the bifurcation set ℓ_N tends to the half line $\mathcal{B}$, the Hamiltonian pitchfork bifurcation set in section 2.

Since the larger N becomes, the larger becomes the value $h + h^2E$ of H_α on Maslov-quantizable invariant tori (see (5)), the half-line obtained above is understood as the classical limit of the bifurcation set for the degeneracy of energy levels in quantum theory. Finally, we arrive at the conclusion:

Conclusion. For the Maslov-quantized perturbed oscillator the Hamiltonian pitchfork bifurcation set $\mathcal{B}$ for the periodic trajectories is the classical limit of the bifurcation set ℓ_N for the degeneracy of energy levels.

6. Concluding remarks

We wish to make mention of the energy level behaviour against the changes in α_2 when α_1 is fixed. We assume again that h is quantized as $h = \hbar(N+1)$. The energy level behaviour against the changes in α_2 is neatly investigated by tracing the line $\{\alpha | \alpha_1 = \text{constant}\}$ in $\mathcal{P}$: since α_1 is now fixed the equality in (44) is thought of as an equation for α_2, so we denote its solution by $\alpha_2^{(N,\nu)}$. For notational convenience we introduce $\alpha_2^{(N,[(N-1)/2]+1)} = \infty$ and $\alpha_2^{(N,-1)} = 0$. Since the middle of (44) can be viewed as a decreasing function of α_2, we see that (44) holds if $\alpha_2^{(N,\nu)} < \alpha_2 \leqslant \alpha_2^{(N,\nu+1)}$. Then, if we start with $\alpha_2 = 0$ and go up along the line $\{\alpha | \alpha_1 = \text{const}\}$, we immediately obtain the following as a corollary of theorem 5.4.

Corollary. Assume that α_1 is fixed and that $h = \hbar(N+1)$. The $\nu + 1$ quantized energy levels are kept doubly-degenerate as far as $\alpha_2 \in [\alpha_2^{(N,\nu)}, \alpha_2^{(N,\nu+1)})$.

The energy level behaviour stated in this corollary deserves comparison with the energy level behaviour in the 'Robnik-quantized' oscillator. Let $\widehat{H}_\alpha$ be a Hamiltonian operator associated with the Hamiltonian H_α through Robnik's scheme (Robnik 1984, Uwano 1994). The eigenvalues of $\widehat{H}_\alpha$ were computed numerically by restricting $\widehat{H}_\alpha$ to the eigenspaces of the harmonic oscillator Hamiltonian $\widehat{J}$ (Uwano 1994). Since the Maslov-quantized energy levels are computed on the torus determined by $E = (H_\alpha - h)/h^2$ with $h = \hbar(N+1)$, it will be better to compare the Maslov-quantized energy levels with the eigenvalues *not* of the Hamiltonian $\widehat{H}_\alpha$ itself *but* of a 'normalized' one, $[\widehat{H}_\alpha - (N+1)\hbar]/\{(N+1)\hbar\}^2$ restricted to the eigenspace of $\widehat{J}$ with the eigenvalue $\hbar(N+1)$. In figure 3 the behaviour of the eigenvalues computed numerically for the normalized operator is described with α_2 varying from 0 to 10 and $\alpha_1 \equiv 1$ (fixed).

From this and further numerical results, three observations have been made about the Robnik-quantized oscillator (Uwano 1994) which are in keeping with proposition 5.2, lemma 5.5, and the corollary of theorem 5.4.

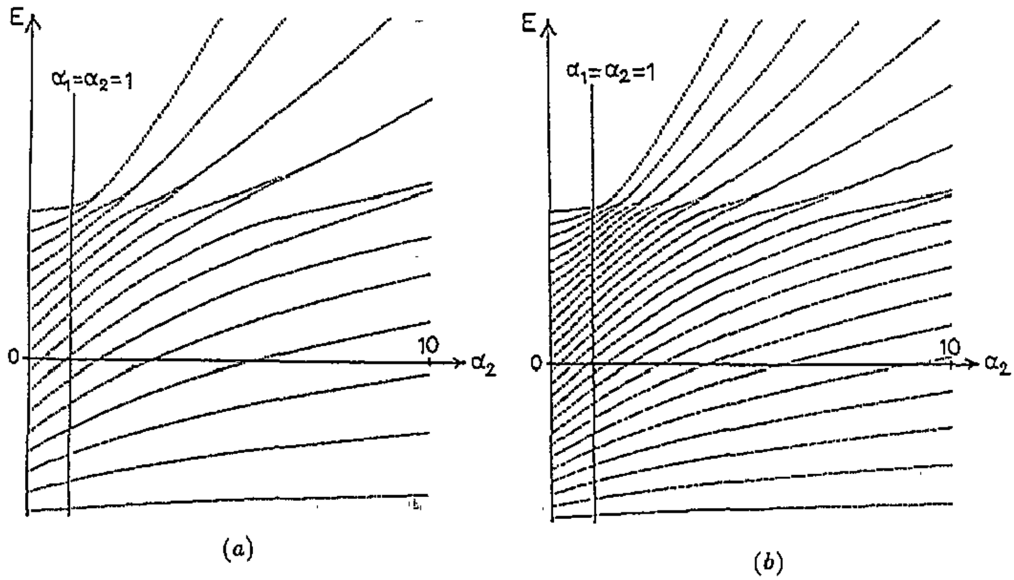

Figure 3. The eigenvalues of $[\widehat{H}_\alpha - \hbar(N+1)]/\{\hbar(N+1)\}^2$. (a) $N = 15$; (b) $N = 25$.

Observations.

(1) The eigenvalues exceeding a certain value are all doubly degenerate.
(2) All degeneracies take place when the inequality $\alpha_1 < \alpha_2$ holds.
(3) For α_1 fixed, once a degeneracy takes place at a certain value of α_2, it is maintained as α_2 increases beyond that value.

For non-resonant Hamiltonians in the Birkhoff–Gustavson normal form the Maslov scheme obviously provides the same energy levels as the Robnik one does (Robnik 1984). However, for resonant Hamiltonians in the normal form like H_α, the agreement in the energy level behaviours is not so trivial. The accounting of the agreement in the behaviour of energy levels will be a future problem.

We wish to make another remark about the Hamiltonian pitchfork bifurcation: the one-dimensional oscillator with double-well potential is well known as a typical dynamical system exhibiting a Hamiltonian pitchfork bifurcation (see Ali and Wood 1989 and references therein). The double well potential also gives rise to degenerate energy levels. However, there exist several differences between the bifurcations of the double-well oscillator and our oscillator: for the double-well oscillator a pitchfork bifurcation occurs at the equilibrium points. In contrast with this, in our perturbed oscillator the bifurcation takes place at periodic trajectories. Further, since our oscillator has two-degrees of freedom, the quantization of it needs the geometry of invariant tori more than the one-dimensional double well oscillator. In fact, the reduction procedure seems to be indispensible to drawing the conclusion that the relation between the bifurcation sets for periodic trajectories and for the degeneracy of energy levels.

Appendix A.

Because of the complete integrability of the perturbed oscillator, the Maslov index for a loop c is given as follows (Arnold 1963, Yoshioka 1986): let M be the complex-valued matrix

$$M = \begin{pmatrix} \dfrac{\partial J}{\partial p_1} & \dfrac{\partial H_\alpha}{\partial p_1} \\ \dfrac{\partial J}{\partial p_2} & \dfrac{\partial H_\alpha}{\partial p_2} \end{pmatrix} + \mathrm{i} \begin{pmatrix} \dfrac{\partial J}{\partial q_1} & \dfrac{\partial H_\alpha}{\partial q_1} \\ \dfrac{\partial J}{\partial q_2} & \dfrac{\partial H_\alpha}{\partial q_2} \end{pmatrix} \tag{A.1}$$

and take the argument $\arg(\det M)$ of its determinant. Then the Maslov index for a loop c is given by the integral

$$\mathcal{M}(c) = \frac{1}{\pi} \oint_c \mathrm{d}\left[\arg(\det(M))\right] \tag{A.2}$$

along c. By a straightforward calculation of (A.2) for the type-I loop $c^{(1)}$ we obtain $\mathcal{M}(c^{(1)}) = 4$. To show (25) for the type-II loop $c^{(2)}$ we make a further evaluation of (A.2). Using equations (2), (7), and (15), we can reduce the integral in (A.2) for $c^{(2)}$ to an integral along the loop $\tilde{c}^{(2)}$ in the (ξ, η)-plane

$$\frac{1}{\pi} \oint_{c^{(2)}} \mathrm{d}[\arg(\det M)] = \frac{1}{\pi} \oint_{\tilde{c}^{(2)}} \mathrm{d}\left[\arg\left\{(\alpha_1 + 2\alpha_2\eta^2)\xi + \mathrm{i}(-\alpha_1 + \alpha_2 - 2\alpha_2\eta^2)\eta\right\}\right]. \tag{A.3}$$

Hence, examining the behaviour of $\tan[\arg\{(\alpha_1 + 2\alpha_2\eta^2)\xi + \mathrm{i}(-\alpha_1 + \alpha_2 - 2\alpha_2\eta^2)\eta\}]$ we obtain $\mathcal{M}(c^{(2)}) = -2$. □

Appendix B.

We derive (31) first. From equation (16) determining the loop $\tilde{c}^{(2)}$ for $c^{(2)}$ we can write down the area integral as

$$\mathrm{Area}\,(\tilde{c}_{+}^{(2)}(\alpha_1;\alpha)) = 2\int_0^{\eta_M} \sqrt{F(\eta)}\,\mathrm{d}\eta \tag{B.1}$$

where $F(\eta)$ is the function in (16), and η_M (> 0) the zero of $F(\eta)$. Then the change of variable $(\xi, \eta) = (\rho\cos\psi, \rho\sin\psi)$ in the right-hand side of (B.1) yields (31). The case for $\tilde{c}_{-}^{(2)}(\alpha_1;\alpha)$ can be proved in a similar way.

We proceed to the elliptic integral expression of the area integral: in a manner similar to that for deriving (B.1) the area integral for the loop $\tilde{c}_{+}^{(2)}(E,\alpha)$ can be written as

$$\mathrm{Area}\,(\tilde{c}_{+}^{(2)}(E,\alpha)) = 2\int_{\eta_m}^{\eta_M} \sqrt{F(\eta)}\,\mathrm{d}\eta \tag{B.2}$$

where η_m and η_M are the zeros of $F(\eta)$ to be $0 < \eta_m < \eta_M$. The change of variables

$$(\xi, \eta) = \left(\frac{\rho w}{\sqrt{1+w^2}}, \frac{\rho}{\sqrt{1+w^2}}\right) \tag{B.3}$$

in the right-hand side of (B.2) yields (32)–(34) for $\tilde{c}_{+}^{(2)}(E,\alpha)$. The area integral for $\tilde{c}_{+}^{(2)}(E,\alpha)$ is thus expressed as the elliptic integral form (32). Through a similar calculation Area $(\tilde{c}_{-}^{(2)}(E,\alpha))$ is shown to be expressed as (32)–(34). For the loop $\tilde{c}_{s}^{(2)}(E,\alpha)$ the area integral takes the form

$$\mathrm{Area}\,(\tilde{c}_{s}^{(2)}(E,\alpha)) = 4\int_0^{\eta_M} \sqrt{F(\eta)}\,\mathrm{d}\eta \tag{B.4}$$

so that through (B.3) it turns out to be expressed as (35).

Appendix C.

Since $F(\eta)$ is differentiable, its zeros η_m, η_M depend smoothly on E. Hence the area integral expressed as (B.2) is differentiated to satisfy

$$\frac{\mathrm{d}}{\mathrm{d}E}\left(\mathrm{Area}\,(\tilde{c}_{+}^{(2)}(E;\alpha))\right) < 0. \tag{C.1}$$

We also obtain (C.1) for $\tilde{c}_{-}^{(2)}(E;\alpha)$ on account of lemma 4.6. Hence the area integral proves to be a decreasing function of E. The continuity of the area integral is now obvious because of its differentiability.

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